Convergence guarantees for Deep Equilibrium models with applications to Poisson inverse problems

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Inverse Problems

$$\boxed{y = \mathcal{N}(Ax)}$$

$A \in \mathbb{R}^{m \times n}_{\geq 0}$
$x \in \mathbb{R}^{n}_{\geq 0}, y \in \mathbb{R}^{m}_{\geq 0}$

Gaussian case: $\mathcal{N}(Ax) = Ax + \eta,\ \eta \sim N(0, \sigma^2 I)$

Poisson case: $\mathcal{N}(Ax) =$ Poisson(Ax)

$$\boxed{\underset{x \in C}{\operatorname{argmin}}\, D(y, Ax) + \lambda g(x)}$$

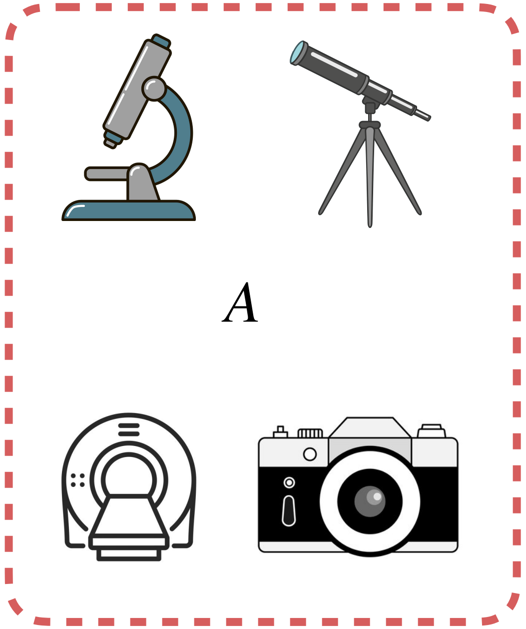
Gaussian case: $D(y, Ax) = \frac{1}{2}\|Ax - y\|_2^2, \quad C = \mathbb{R}^n$

Poisson case: $D(y, Ax) =$ KL$(y, Ax), \quad C = \mathbb{R}^n_+$

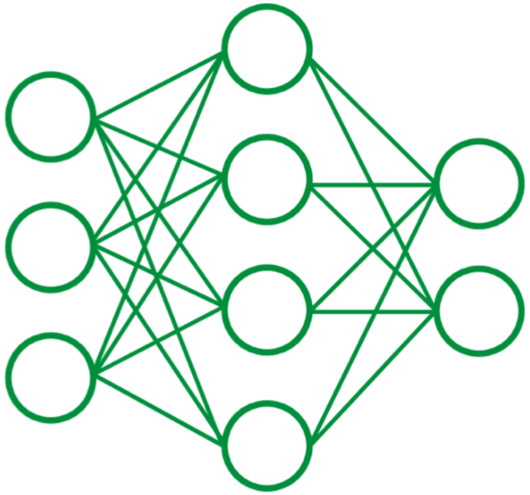
$$g(x) = \begin{cases} \|x\|_2^2 & \text{Tikhonov} \\ \|x\|_1 & \text{L1} \\ \text{TV}(x) & \text{Total Variation} \end{cases}$$

Hybrid data-driven approach

Physics



$+$



Goal:
given $\underset{x\in C}{\operatorname{argmin}}\, D(y, Ax) + \lambda g(x)$, learning λg

Data-driven approaches for IPs

- PnP

[Venkatakrishnan, Bouman, Wohlberg, '13]

$$x^{k+1} = \overset{\mathscr{D}}{\cancel{\mathrm{Prox}_{\tau g}}}(x^k - \tau \nabla D(y, Ax^k))$$

- Unrolling

[Monga, Li and Eldar '21]

$$x^{k+1} = F(x^k, y, A, \theta^k),\ k = 1,\ldots,K, \quad K < +\infty$$

- Deep equilibrium models

$$x^0 \longrightarrow f_\theta \circ f_\theta \circ \ldots \circ f_\theta \circ \ldots \longrightarrow x^\infty$$

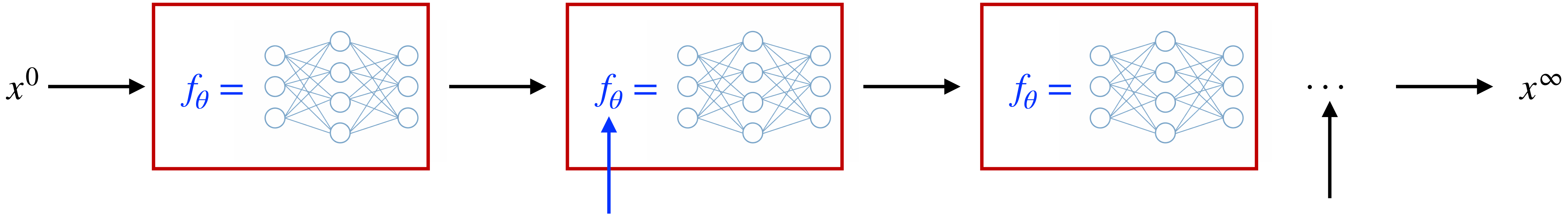
[Bai, Kolter, Koltun '19]

[Gilton, Ongie, Willett '21]

[Zou, Liu, Wohlberg and Kamilov '23]

Deep Equilibrium models (DEQs) [Bai, Kolter and Koltun '19, Gilton, Ongie and Willett '21]

Input: $y \in Y$

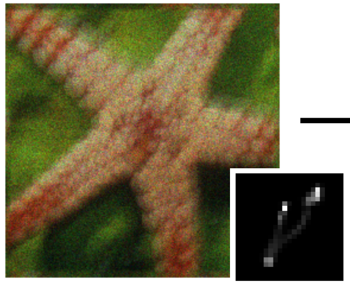


$f_\theta(\,\cdot\,; y)$ DEQ layer

Infinite composition

$$y \mapsto \mathrm{Fix}(f_\theta(\,\cdot\,; y)) \quad \text{DEQ}$$

y



$\xrightarrow{f_\theta(\,\cdot\,; y)}$

x^∞

How to design a well-posed DEQ

$y \in Y \subseteq \mathbb{R}^m$, $f_\theta(\cdot, y): X \subseteq \mathbb{R}^n \to X$

Well-posed DEQ

$\forall x^0 \in X,$ the sequence $\{x^k\}_{k\geq 0},$ given by $x^{k+1} = f_\theta(x^k, y)$ converges to $x^* \in \mathrm{Fix}(f_\theta(\cdot, y))$

Sufficient condition (contraction):

$$\exists\ \rho < 1 \text{ s.t. } \|f_\theta(x_1, y) - f_\theta(x_2, y)\| \leq \rho\|x_1 - x_2\|\ \forall x_1, x_2 \in X$$

Then: $\mathrm{Fix}(f_\theta(\cdot, y)) = \{x^*\}$ and $\forall x^0 \in X$, $x^k \to x^*$

First-order optimisation contractive operators

Gradient Descent (GD) [Briceño-Arias, Pustelnik '23]

Let $F = D(y, A\cdot) + \lambda g$ be L_F-smooth

$$\mathrm{GD}(x) = x - \tau \nabla F(x) = (I - \tau \nabla F)(x)$$

If F is μ_F-strongly convex and $\tau \in (0,2/L_F)$, then GD **is a contraction** with:

$$\rho = \max\{\,|1 - \tau\mu_F|\,,|1 - \tau L_F|\,\} < 1$$

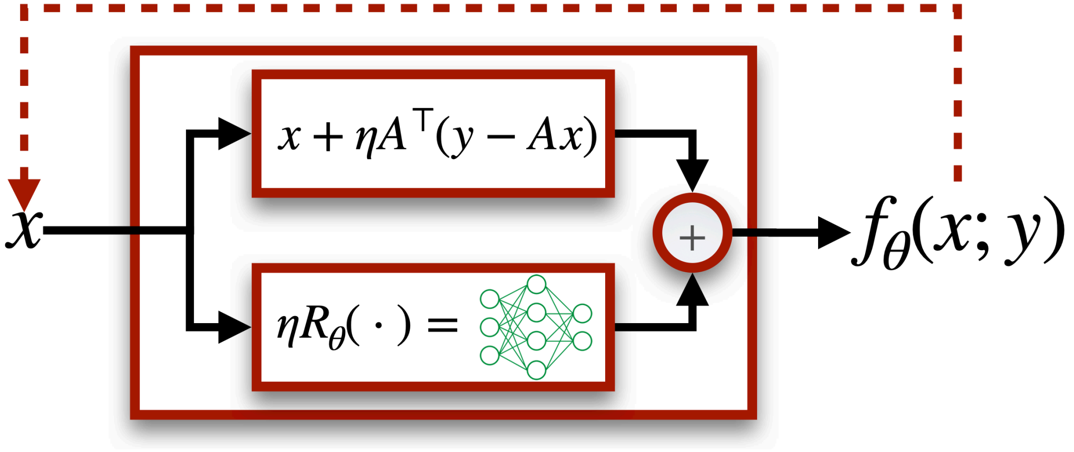
Proximal Gradient Descent (PGD) [Taylor, Hendrickx, Glineur '18]

Let $F = D(y, A\cdot) + \lambda g$ with $D(y, A\cdot)$ L_D-smooth and g convex/non-smooth

$$\mathrm{PGD}(x) = \mathrm{prox}_{\tau g}(x - \tau \nabla D(y, Ax))$$

If $D(y, A\cdot)$ is μ_D-strongly convex, then PGD **is a contraction** for suitable choices of τ

Contractive-like DEQ: DE-GRAD



$$D(y, Ax) = \frac{1}{2}\|Ax - y\|_2^2$$

$\lambda \nabla g \to R_\theta$ parameterise the gradient

$$f_\theta(x; y) = (I - \tau \nabla D(y, A\cdot))(x) - \tau R_\theta(x)$$

Deep Equilibrium Gradient Descent (DE-GRAD) [Gilton, Ongie and Willett '21]

$$\|f_\theta(x_1; y) - f_\theta(x_2; y)\| \leq \|(I - \tau \nabla D(y, A\cdot))(x_1 - x_2)\| + \tau\|R_\theta(x_1) + R_\theta(x_2)\|$$

If the following holds:

- $D(y, A\cdot)$ is L_D-smooth and μ_d- strongly convex
- R_θ is $L_{R_\theta}-$Lipschitz

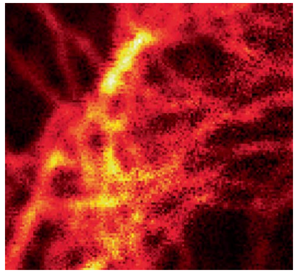
then GD$_\theta$ is a contraction $\forall \tau \in (0, 2/(L_D + L_{R_\theta}))$

Poisson Inverse Problems case

$$\boxed{y = \text{Poisson}(Ax)}$$

$A \in \mathbb{R}^{m\times n}_{\geq 0}$
$x \in \mathbb{R}^{n}_{\geq 0}, y \in \mathbb{R}^{m}_{\geq 0}$

Motivation: Imaging with low photon count (e.g. Microscopy) $\longrightarrow$



[Bertero and Boccacci '98]

$$\boxed{\underset{x\geq 0}{\text{argmin}}\, \mathsf{KL}(y, Ax) + \lambda g(x)}$$

$$\mathsf{KL}(y, Ax) = \sum_{i=1}^{m} d_{\mathsf{KL}}(y_i, (Ax)_i)$$

$$d_{\mathsf{KL}}(y, x) = y \log\left(\frac{y}{x}\right) + x - y$$

- ≥ 0
- convex
- **not L-smooth**

$\longrightarrow$ GD not applicable

$\lambda g \to R_\theta$ parameterise the regularisation function

KL minimisation: Mirror Descent

$$\boxed{\operatorname*{argmin}_{x\geq 0} \mathsf{KL}(y, Ax) + R_\theta(x) + \iota_{[0,a]^n} := \Psi_\theta(\,\cdot\,; y)}$$

$$h = \frac{1}{2}\|x\|_2^2 \implies$$

$$D_h(x,y) = \frac{1}{2}\|x-y\|_2^2$$

Given a Legendre function h:

$$x^{k+1} = \operatorname*{argmin}_{x\in\mathbb{R}^n} \langle x - x^k, \nabla(\mathsf{KL}(y, A\cdot) + R_\theta)(x^k)\rangle + \frac{1}{\tau} D_h(x, x^k) + \iota_{[0,a]^n}(x)$$

Burg's entropy: $h(x) = -\sum_{i=1}^{n} \log(x_i), \quad \mathrm{dom}(h) = (0, +\infty)^n$

[Bauschke, Bolte and Teboulle '17]

Key points:

$x^0 \in \mathrm{int}(\mathrm{dom}(h)) \implies x^k \in \mathrm{int}(\mathrm{dom}(h))\,\forall k \geq 0 \longrightarrow$ no need of projection

KL minimisation: Mirror Descent

$$\boxed{\underset{x\geq 0}{\operatorname{argmin}}\ \mathrm{KL}(y, Ax) + R_\theta(x) + \iota_{[0,a]^n} := \Psi_\theta(\,\cdot\,; y)}$$

$$h = \frac{1}{2}\|x\|_2^2 \implies$$

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Burg's entropy: $h(x) = -\sum_{i=1}^{n} \log(x_i), \quad \mathrm{dom}(h) = (0, +\infty)^n$ [Bauschke, Bolte and Teboulle '17]

Key points:

$\tau \propto \frac{1}{L}$, $L > 0$ is s.t. $Lh - (\mathrm{KL} + R_\theta)$ is convex on $\mathrm{int}(\mathrm{dom}(h))$

NoLip property [Bauschke, Bolte and Teboulle '17]

L-SMAD [Bolte, Sabach, Teboulle and Vaisbourd '18]

Relative smoothness [Lu, Freund and Nesterov '18]

If $h = \frac{1}{2}\|x\|_2^2 \implies L$-smoothness

Deep equilibrium Mirror Descent: DEQ-MD

$$f_\theta(x;y) = \Pi_{[0,a]^n}(\nabla h^*(\nabla h(x) - \tau(\nabla \mathsf{KL}(y, Ax) + \nabla R_\theta(x))))$$

Well-posedness of DEQ-MD $\Longleftrightarrow$ convergence iterates of MD to minimise Ψ_θ

Ψ_θ non-smooth and non-convex functional, convergence relies on [Bolte, Sabach, Teboulle and Vaisbourd '18]:

- **Kurdyka-Łojasiewicz property**
- MD provides a bounded *gradient-like descent* sequence ($\Pi_{[0,a]^n}$ + relative smoothness)

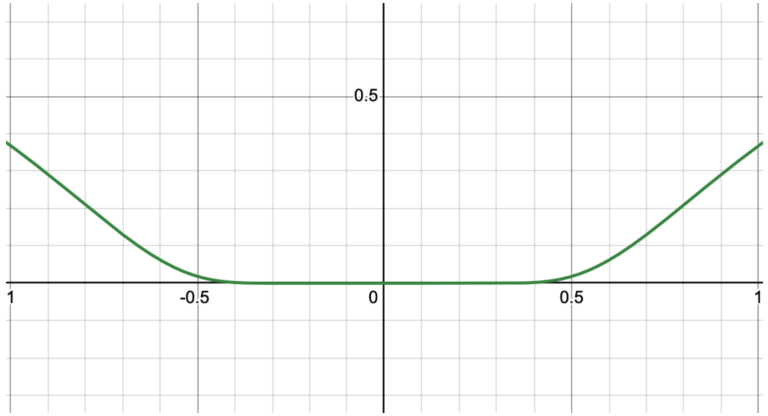
Choose R_θ s.t. Ψ_θ is a KŁ function

Kurdyka-Łojasiewicz property [Attouche, Bolte and Svaiter '13]

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have the **Kurdyka-Łojasiewicz property** at $x^* \in \mathrm{dom}(f)$ if there exists a **neighborhood** U of x^* and a continuous concave function $\psi : [0,\delta) \to \mathbb{R}_+$ such that:
$\forall x \in U \cap \{x \mid f(x^*) < f(x) < f(x^*) + \delta\}$, the Kurdyka-Łojasiewicz inequality holds:

$$\psi'(f(x) - f(x^*))\mathrm{dist}(0, \partial f(x)) \geq 1.$$

Idea: *The functional has to be sufficiently sharp around critical points (upon reparameterisation)*



Counterexample: $f(x) = e^{-1/x^2}$

But it is satisfied by a wide class of functions: Analytical, Semialgebraic, Subanalytical...

Ψ_θ is a Kurdyka-Łojasiewicz function

Assumption: R_θ is analytic on $\mathbb{R}^n$

$$\Psi_\theta = \underbrace{\mathsf{KL}(y, Ax)}_{\textit{analytical}} + \underbrace{R_\theta}_{\textit{analytical}} + \underbrace{\iota_{[0,a]^n}}_{\textit{semialgebraic}}$$

[Łojasiewicz '99]

[Łojasiewicz '99]

subanalytical
(stable to sum)

Ψ_θ is a KŁ function

[Bolte, Danilidis and Lewis '07]

subanalytic functions with **closed domain** and continuous upon restriction are KŁ

e.g. $A = I$ (denoising)

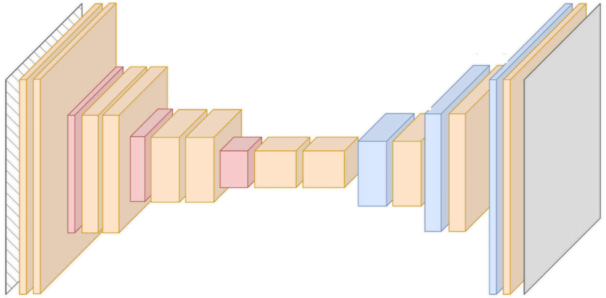
$\implies$ $\mathrm{dom}(\Psi_\theta)$ **not closed**

$A = I \implies \mathrm{dom}(\Psi) = (0,a]^n$

Numerical implementation

Choice of R_θ:

$$R_\theta(x) = \frac{1}{2}\|x - N_\theta(x)\|^2, N_\theta =$$



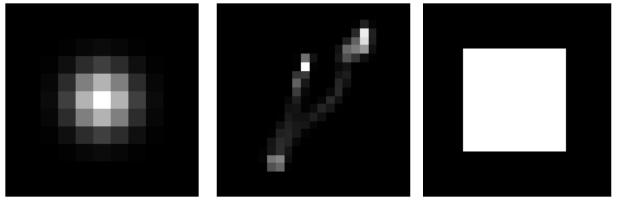
[Romano, Elad, and Milanfar '17]
[Hurault, Leclaire and Papadakis '21]

Dataset:
500 natural images of size 256 x 256

Data generation:

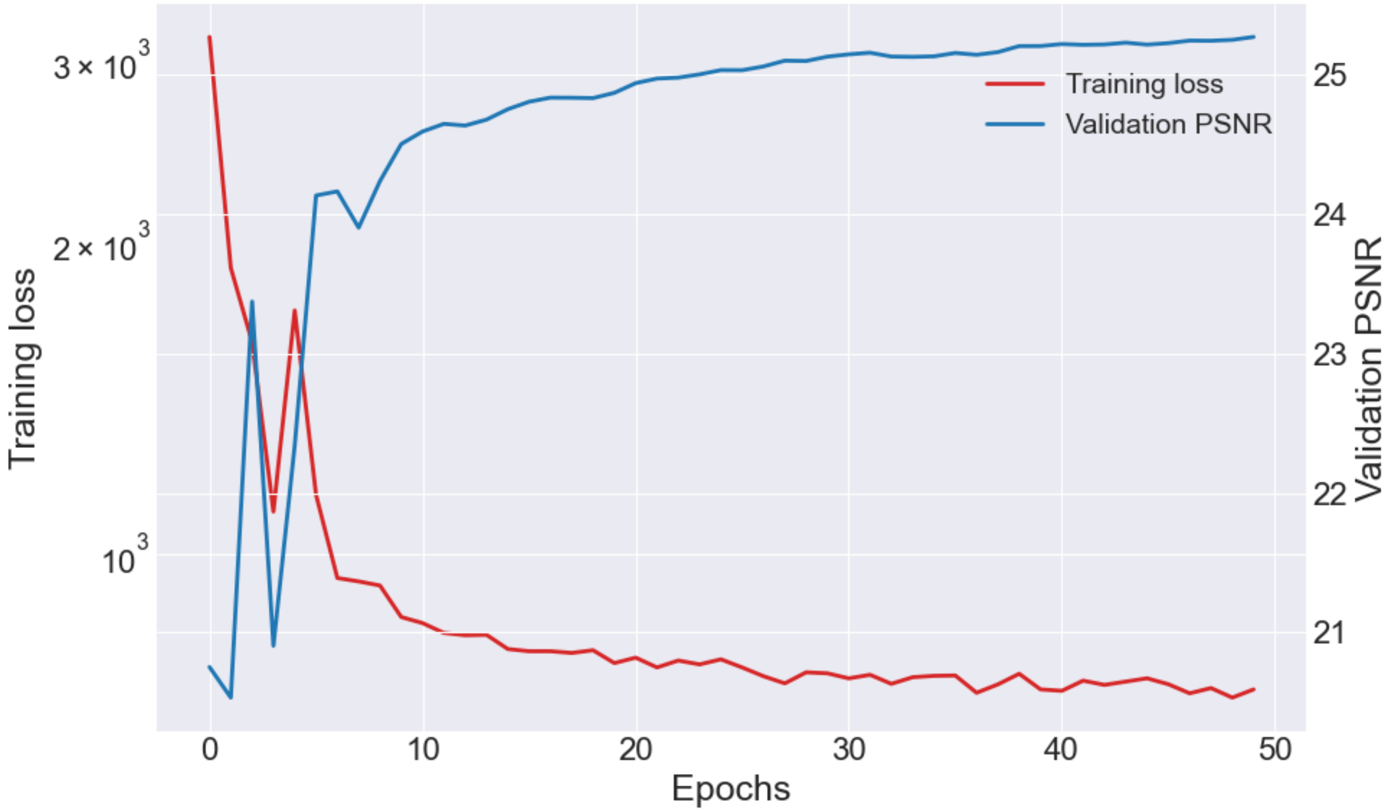
$$y = \text{Poiss}(\alpha A x)$$

$$A =$$



Loss function:

$$\ell(x^\infty, x) = \|x^\infty - x\|^2$$



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Ψ_θ is a Kurdyka-Łojasiewicz function

Proposition [D., Villa, Vaiter, Calatroni '25]

Subanalytic functions Ψ, which are **l.s.c., coercive**, **continuous** whenever restricted to their domain and such that:

$$S := \{x \in \overline{\mathrm{dom}(\Psi)} \mid \Psi(x) = +\infty\}$$

is closed, are KŁ.

Example: $n = 2,\ A = I, \mathrm{dom}(\Psi_\theta) = (0,a]^2,\ S = ([0,a] \times \{0\}) \cup (\{0\} \times [0,a])$

R_θ analytic $\Longrightarrow$ DEQ-MD is well-posed

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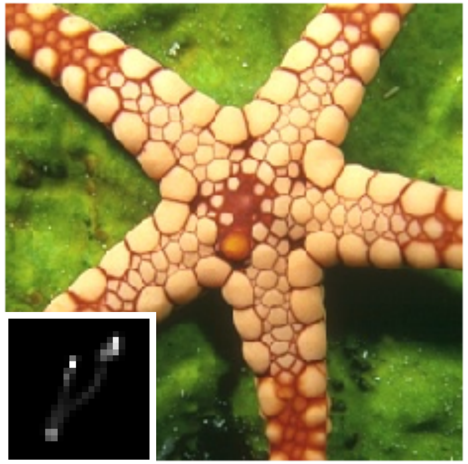
Numerical tests:

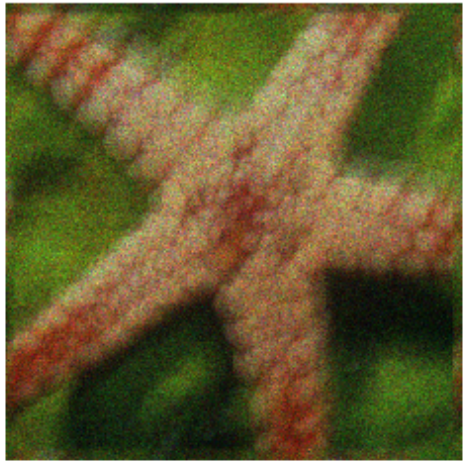
$\alpha = 60$

Ground truth



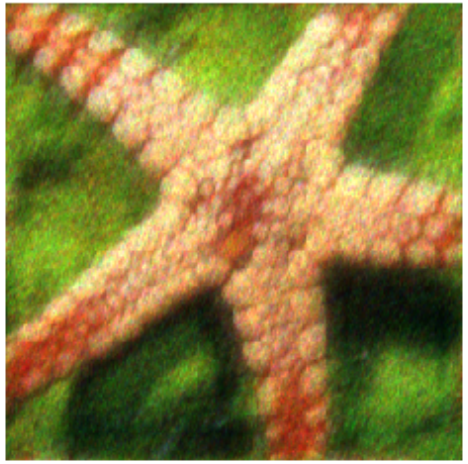
PSNR ↑, SSIM ↑

Measurement

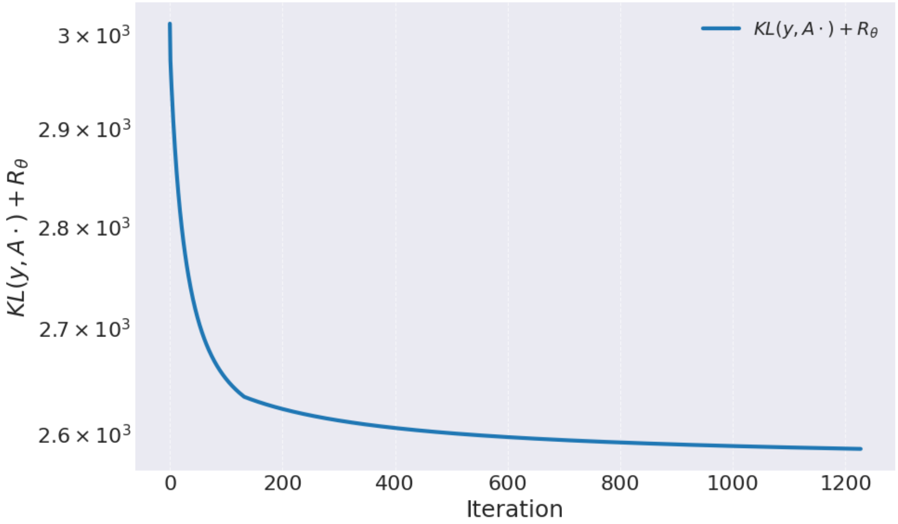


13.3058, 0.1664

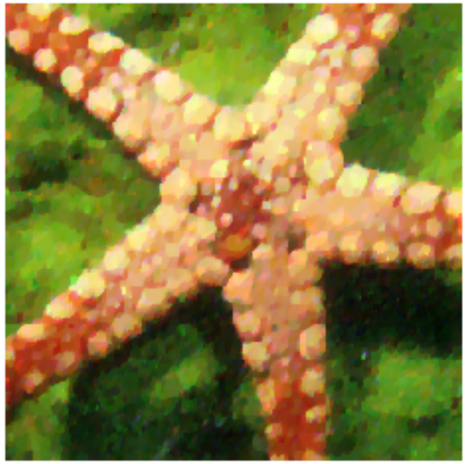
Richardson-Lucy



20.4583, 0.4737



Total Variation



22.3863, 0.5817

B-RED

<u>24.2011, 0.7044</u>

DEQ-RED

24.6214, 0.7255

	$\alpha = 100$			$\alpha = 60$		
Method	PSNR ↑	SSIM ↑	LPIPS ↓	PSNR ↑	SSIM ↑	LPIPS ↓
RL	22.7381	0.5379	0.5570	22.2968	0.5111	0.6017
TV+KL	23.9879	0.6304	<u>0.3888</u>	23.5203	0.6045	<u>0.4254</u>
B-RED	<u>25.0710</u>	<u>0.6939</u>	0.4120	**24.6566**	**0.6723**	0.4384
B-PnP	24.8976	0.6847	0.4417	24.5818	<u>0.6674</u>	0.4401
DEQ-S	24.2080	0.6730	0.3935	23.7252	0.6221	0.4346
DEQ-RED	**25.0955**	**0.6958**	**0.3317**	<u>24.5940</u>	0.6650	**0.3755**

Conclusions & outlook

- DEQ-MD as a convergent learning-the-regularisation framework

  $\longrightarrow$ Bregman MM? [Adly, Chazottes, Chouzenoux, Pesquet, Sureau, '26]

- Convergence beyond L-smoothness and via refined KŁ-like argument

  $\longrightarrow$ General result: other applications?

- Application to Poisson IPs

Thanks for your attention!

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- C. Daniele, S. Villa, S. Vaiter, L. Calatroni, *Deep equilibrium models for Poisson Imaging Inverse Problems via Mirror Descent*, to appear in **SIAM Journal of Imaging Sciences**, 19 (2), (2026).
- C. Daniele, S. Villa, S. Vaiter, L. Calatroni, *Noise and task-adaptive Deep-Equilibrium Mirror-Descent for Poisson Image reconstruction*, submitted (2026).

Forward pass

Challenges:

Fixed point iteration $x^{k+1} = f_\theta(x^k; y)$ slow (sublinear rate)

$\tau \propto \dfrac{1}{L}$ with $L > 0$ s.t. $Lh - (\mathsf{KL} + R_\theta)$ is convex, hard to compute

Backtracking for DEQ-MD [Hurault, Kamilov, Leclaire and Papadakis '23]

Input: $\gamma \in (0,1), \eta \in (0,1)$
Define: $T_\tau(x) = \nabla h^*(\nabla h(x) - \tau(\nabla \mathsf{KL}(y, Ax) + \nabla R_\theta(x))$
while $\Psi_\theta(x^k) - \Psi_\theta(T_\tau(x^k)) < \dfrac{\gamma}{\tau} D_h(T_\tau(x^k), x^k)$ **do**
$\tau \leftarrow \eta\tau$

Update: $x^{k+1} = T_\tau(x^k)$

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Backward pass:

$\{(x_i, y_i)\}_{i=1}^N,\ y_i = \text{Poisson}(Ax_i), \ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

$$\text{find } \hat{\theta} \in \underset{\theta \in \Theta}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} \ell(x^\infty(\theta; y_i, x_i^0), x_i)$$

$$\bar{\theta} \in \Theta, \nabla_\theta \ell(x^\infty(\bar{\theta}; y, x^0), x) = \frac{\partial x^\infty(\bar{\theta}; y, x^0)}{\partial \theta}^\top \frac{\partial \ell(x^\infty(\bar{\theta}; y, x^0))}{\partial x} \qquad \frac{\partial x^\infty(\bar{\theta}; y, x^0)}{\partial \theta} = \frac{f_{\bar{\theta}}(x^\infty(\bar{\theta}; y\,.\,x^0))}{\partial \theta}$$

$$\frac{\partial x^\infty(\bar{\theta}; y, x^0)}{\partial \theta} = \left(I - \frac{\partial f_{\bar{\theta}}(x)}{\partial x} \bigg|_{x = x^\infty} \right)^{-1} \frac{\partial f_{\bar{\theta}}(x^\infty(\bar{\theta}; y, x^0))}{\partial \theta}$$

Constant memory

Backward pass:

$\{(x_i, y_i)\}_{i=1}^N,\ y_i = \text{Poisson}(Ax_i), \ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

$$\boxed{\text{find } \hat{\theta} \in \underset{\theta \in \Theta}{\text{argmin}} \frac{1}{N} \sum_{i=1}^{N} \ell(x^\infty(\theta; y_i, x_i^0), x_i)}$$

$$\bar{\theta} \in \Theta, \nabla_\theta \ell(x^\infty(\bar{\theta}; y, x^0), x) = \frac{\partial x^\infty(\bar{\theta}; y, x^0)}{\partial \theta}^\top \frac{\partial \ell(x^\infty(\bar{\theta}; y, x^0))}{\partial x}$$

$$\frac{\partial x^\infty(\bar{\theta}; y, x^0)}{\partial \theta} = \frac{f_{\bar{\theta}}(x^\infty(\bar{\theta}; y\,.\,x^0))}{\partial \theta}$$

$$\boxed{\frac{\partial x^\infty(\bar{\theta}; y, x^0)}{\partial \theta} \approx \left(I - \cancel{\frac{\partial f_{\bar{\theta}}}{\partial x}\Big|_{x^\infty}} \right)^{-1} \frac{\partial f_{\bar{\theta}}(x^\infty(\bar{\theta}; y, x^0))}{\partial \theta}}$$

Jacobian Free Backpropagation/
One step differentiation

[Fung, Heaton, Li, McKenzie, Osher and Yin '22]
[Bolte, Pauwels and Vaiter '23]

$$\|\frac{\partial f_{\bar{\theta}}}{\partial x}\| < 1$$

Our case: $\|\frac{\partial f_{\bar{\theta}}}{\partial x}\| \leq 1$

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Convergence proof structure: idea

- **Issue:** classical KŁ results require a *closed domain*, but $\text{dom}(\Psi)$ is not closed.

Idea: isolate region around $\mathbf{x}^* \in \text{Crit}(\Psi)$ avoiding $+\infty$

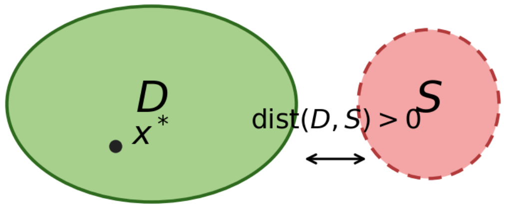
- **Step 1**: localise around $\mathbf{x}^* \in \text{Crit}(\Psi)$ considering $D = \Psi^{-1}(-\infty, M]$, where $M = \Psi(\mathbf{x}^*)$.
- $D \in \mathbb{R}^n$ is compact and separated from $S = \left\{\mathbf{x} : \Psi(\mathbf{x}) = +\infty\right\}$? Then $\varepsilon = \text{dist}(D, S) > 0$
- $\mathbf{x}^* \in \text{int}(D)$, set $D' = D$.
- If $\mathbf{x}^* \in \partial D$, enlarge slightly to D_ε and define $B_\varepsilon = D_\varepsilon \cap \text{dom}(\Psi) \to$ find $D' = \Psi(-\infty, M_\varepsilon], M_\varepsilon < +\infty$
- **Step 2:** Having $D' \subset \text{dom}(\Psi)$, construct $U(\mathbf{x}^*)$: $U(\mathbf{x}^*) \cap \text{dom}(\Psi) \subset D'$

Idea: apply classical results

- D' is subanalytic.
- **Step 3:** Define $\tilde{\Psi} = \Psi + \iota_{D'}$. It is subanalytic with closed domain. Theory applies.
- KŁ holds for $\tilde{\Psi}$ at $\mathbf{x}^*$
- **Conclusion:** $\Psi \equiv \tilde{\Psi}$ on $U(\mathbf{x}^*)$

Convergence proof structure: idea

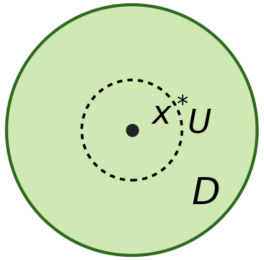
1. Geometry



$D = \Psi^{-1}((-\infty, M])$

$S = \left\{ x \in \overline{\mathrm{dom}(\Psi)} : \Psi(x) = +\infty \right\}$

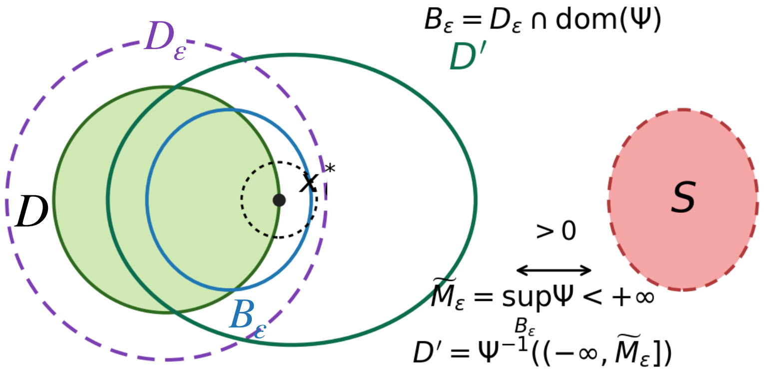
2a. Case $x^* \in \mathrm{int}(D)$



Take $D' = D$.

$U \cap \mathrm{dom}(\Psi) \subset D'$.

2b. Case $x^* \in \partial D$



$U \cap \mathrm{dom}(\Psi) \subset D'$

3. Reduction

$$\widetilde{\Psi} = \Psi + \iota_{D'}$$

$\mathrm{dom}(\widetilde{\Psi}) = D'$ is closed

D' is subanalytic as

$D' = \Psi^{-1}([-m, M])$

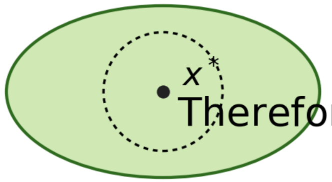
$\tilde{\Psi}$ is subanalytic

4. Apply classical KL

Apply the classical KL theorem to $\widetilde{\Psi}$ at x^*.

$\tilde{\Psi}$ satisfies KŁ at x^*.

5. Transfer back locally



On U, one has $\widetilde{\Psi} = \Psi$.

Therefore Ψ satisfies the KL property at x^*.